

# Quark number susceptibilities from HTL-resummed thermodynamics

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## Abstract

We compute analytically the diagonal quark number susceptibilities for a quark-gluon plasma at finite temperature and zero chemical potential, and compare with recent lattice results. The calculation uses the approximately self-consistent resummation of hard thermal and dense loops that we have developed previously. For temperatures between  $1.5$  to  $5T_c$ , our results follow the same trend as the lattice data, but exceed them in magnitude by about  $5 - 10\%$ . We also compute the lowest order contribution, of order  $\alpha_s^3 \log(1/\alpha_s)$ , to the off-diagonal susceptibility. This contribution, which is not a part of our self-consistent calculation, is numerically small, but not small enough to be compatible with a recent lattice simulation.

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## 1 Introduction

A lot of effort is presently devoted to understanding the properties of hot and dense matter from Quantum Chromodynamics. This is motivated in part by the ongoing experimental program on ultrarelativistic heavy ion collisions, and also by the progress in lattice gauge calculations which provide so far the best theoretical tool at our disposal to calculate from first principles the properties of the quark-gluon plasma. Recently however, it has been shown that results of such calculations could be accurately reproduced by weak coupling techniques when the temperature is larger than 2 to 3 times the transition temperature [1,2,3]. The purpose of this paper is to apply these techniques to the calculation of quark-number susceptibilities which have received recently considerable attention.

These quantities are interesting in several respects. First of all, they are to date about the only quantities that can be calculated on the lattice and provide information about finite density [4,5,6,7,8,9]. (Recall that lattice calculations are still limited to zero chemical potential; susceptibilities involve derivatives of the thermodynamic functions with respect to  $\mu$ , and their limit as  $\mu \rightarrow 0$  can be computed on the lattice.) Susceptibilities have also been discussed lately in the context of heavy ion collisions, as they can be related to measurable fluctuations in conserved quantities [10,11,12]. However, the main question addressed here is a theoretical one, namely, whether the recent lattice results in Refs. [8,9] can be explained within resummed perturbation theory, that is, without invoking genuine non-perturbative contributions.

The lattice results [8,9] for the diagonal susceptibility  $\chi$  (cf. eqs. (2)–(3) below) at temperatures between  $1.5$  and  $5T_c$  show a slow approach of the ideal-gas result from below, with deviations of about 15%. But the weak coupling expansion of  $\chi$  completely fails to reproduce this behaviour. In massless QCD at  $\mu = 0$ , this expansion is presently known to order  $g^4 \log(1/g)$  [13,14] :

$$\begin{aligned} \frac{\chi}{\chi_0} = 1 - \frac{1}{2} \frac{3}{N} \frac{N_g}{8} \left(\frac{g}{\pi}\right)^2 + \frac{3}{N} \frac{N_g}{8} \sqrt{\frac{N}{3} + \frac{N_f}{6}} \left(\frac{g}{\pi}\right)^3 \\ + \frac{3}{4} \frac{N_g}{8} \left(\frac{g}{\pi}\right)^4 \log \frac{1}{g} + \mathcal{O}(g^4) \end{aligned} \quad (1)$$

(with  $\chi_0 = NT^2/3$  the ideal gas value and  $N_g = N^2 - 1$ ). Leaving aside the still undetermined  $g^4$  contribution, one finds that the perturbative results lie *above* the ideal gas values for all temperatures of interest, and decrease with increasing  $T$ .

We may relate this failure to that encountered in the perturbative calculation of the pressure [15]. In both cases, the difficulty of perturbation theory has its origin in collective phenomena which develop at the scale  $gT$ , and which in a strict perturbative expansion provide large contributions starting at order  $g^3$ . To cope with this, various resummation schemes have been proposed [1,2,3,16,17,18]. Here, we shall use the one developed in Refs. [1,2,3], which focuses on the physical picture of the quark-gluon plasma as a gas of quasiparticles with properties determined by the “hard thermal loops” (HTL) [19,20]. This approach has proven to be successful in describing the lattice data for the thermodynamics of QCD down to temperatures as low as  $2.5T_c$ .

First lattice measurements of the *off-diagonal* susceptibility  $\chi_{ud}$  have also been reported in Ref. [9]. This quantity vanishes for the ideal gas, so it probes directly the interactions in the system. We show that, when  $\mu = 0$ , it is of order  $g^6 \log(1/g)$ , and we compute this lowest-order contribution in both QCD and QED.

## 2 Diagonal susceptibilities

Quark number susceptibilities are generally defined as:

$$\chi_{ij} \equiv \frac{\partial \mathcal{N}_i}{\partial \mu_j} = \frac{\partial^2 P}{\partial \mu_i \partial \mu_j} = \chi_{ji} \quad (2)$$

where  $i, j$  are flavor indices,  $\mathcal{N}_i$  is the quark number density, and  $P$  is the pressure. With all quarks massless and  $\mu_i = 0$  (as appropriate for comparison with the lattice results), all diagonal and all off-diagonal elements become equal, and we write

$$\chi_{ij}\big|_{\mu=0} \equiv \chi \quad \text{for } i = j, \quad \chi_{ij}\big|_{\mu=0} \equiv \tilde{\chi} \quad \text{for } i \neq j. \quad (3)$$

We shall evaluate the diagonal susceptibility  $\chi$  within the resummation scheme developed in Refs. [1,2,3]. This is based on the following expression for the fermion number density  $\mathcal{N}$  in terms of the dressed fermion propagators  $\Delta_{\pm}$  (cf. eqs. (4.12) and (4.19) of Ref. [3]):

$$\begin{aligned} \mathcal{N} = -4N \int \frac{d^4 k}{(2\pi)^4} \frac{\partial f(\omega)}{\partial \mu} \big\{ & \text{Im} \log \Delta_+^{-1} + \text{Im} \log(-\Delta_-^{-1}) + \\ & - \text{Im} \Sigma_+ \text{Re} \Delta_+ + \text{Im} \Sigma_- \text{Re} \Delta_- \big\}, \end{aligned} \quad (4)$$

where  $\Delta_{\pm}^{-1} \equiv -[\omega \mp (k + \Sigma_{\pm})]$ ,  $\Sigma_{\pm}$  are the corresponding self-energies, and the plus (minus) subscript applies to fermions whose chirality is equal (opposite) to their helicity. The fermion self-energies and propagators are diagonal in flavor indices, and eq. (4) applies to each quark flavor  $i$  separately, but flavor indices are kept implicit.

As in Refs. [1,2,3], we shall consider two successive approximations to the self-energies  $\Sigma_{\pm}$ . The first is the HTL approximation where [19]:

$$\hat{\Sigma}_{\pm}(\omega, k) = \frac{\hat{M}^2}{k} \left( 1 - \frac{\omega \mp k}{2k} \log \frac{\omega + k}{\omega - k} \right), \quad (5)$$

and  $\hat{M}^2$  is the plasma frequency for fermions, i.e., the frequency of long-wavelength ( $k \rightarrow 0$ ) fermionic excitations ( $C_f = (N^2 - 1)/2N$ ):

$$\hat{M}^2 = \frac{g^2 C_f}{4\pi^2} \int_0^{\infty} dk k (2n(k) + f_+(k) + f_-(k)) = \frac{g^2 C_f}{8} \left( T^2 + \frac{\mu^2}{\pi^2} \right). \quad (6)$$

In this approximation, there is no mixing between quarks of different flavors, so the corresponding susceptibilities are diagonal even for  $\mu \neq 0$ .

The resulting expression of the number density, denoted by  $\mathcal{N}_{HTL}$ , is the sum of two contributions:  $\mathcal{N}_{HTL} = \mathcal{N}_{HTL}^{\text{QP}} + \mathcal{N}_{HTL}^{\text{LD}}$ , where  $\mathcal{N}_{HTL}^{\text{QP}}$  is the contribution of the quasiparticle poles<sup>1</sup>  $\omega_{\pm} = \pm[k + \hat{\Sigma}_{\pm}(\omega_{\pm}, k)]$ , and  $\mathcal{N}_{HTL}^{\text{LD}}$  is that of the Landau damping cuts at  $-k < \omega < k$ . We have:

$$\begin{aligned} \mathcal{N}_{HTL}^{\text{QP}} = N \int_0^{\infty} \frac{k^2 dk}{\pi^2} \frac{\partial}{\partial \mu} \left\{ T \log(1 + e^{-(\omega_+(k) - \mu)/T}) \right. \\ \left. + T \log \frac{1 + e^{-(\omega_-(k) - \mu)/T}}{1 + e^{-(k - \mu)/T}} + (\mu \rightarrow -\mu) \right\}, \end{aligned} \quad (7)$$

where the  $\mu$  derivative is to be applied to the explicit  $\mu$  dependence only, and not to that implicit in the dispersion laws  $\omega_+(k)$  and  $\omega_-(k)$  of the quasiparticles. Similarly,

$$\begin{aligned} \mathcal{N}_{HTL}^{\text{LD}} = -N \int_0^{\infty} \frac{k^2 dk}{\pi^3} \int_0^k d\omega \left[ \frac{\partial f_+(\omega)}{\partial \mu} + \frac{\partial f_-(\omega)}{\partial \mu} \right] \\ \times \left\{ \arg[k - \omega + \hat{\Sigma}_+(\omega, k)] - \text{Im } \hat{\Sigma}_+(\omega, k) \text{Re}[k - \omega + \hat{\Sigma}_+(\omega, k)]^{-1} \right. \\ \left. + \arg[k + \omega + \hat{\Sigma}_-(\omega, k)] - \text{Im } \hat{\Sigma}_-(\omega, k) \text{Re}[k + \omega + \hat{\Sigma}_-(\omega, k)]^{-1} \right\}. \end{aligned} \quad (8)$$

The HTL approximation [19] contains the perturbative contributions of order  $g^2$ . This comes exclusively from the hard ( $k \sim T$ ), “normal”, branch  $\omega_+$  and its asymptotic thermal mass  $M_{\infty}^2 \equiv 2k\hat{\Sigma}_+(\omega = k) = 2\hat{M}^2$ :

$$\mathcal{N}^{(2)} = -\frac{N}{2\pi^2} \mu M_{\infty}^2. \quad (9)$$

However, there is no  $g^3$  contribution in  $\mathcal{N}_{HTL}$ . Such a contribution, denoted as  $\mathcal{N}^{(3)}$ , comes entirely from the next-to-leading (NLO) correction  $\delta M_{\infty}^2(k) \equiv 2k \text{Re } \delta \Sigma_+(\omega = k)$  to the asymptotic mass of the hard fermion [2,3]:

$$\mathcal{N}^{(3)} = -4N \int \frac{d^3 k}{(2\pi)^3 2k} \frac{\partial f_+(k)}{\partial \mu} \text{Re } 2k \delta \Sigma_+(\omega = k), \quad (10)$$

where the NLO self-energy  $\delta \Sigma_+$  is given by the diagrams in Fig. 1.

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<sup>1</sup> Charge conjugation ( $\Sigma_+(\omega, k) = \Sigma_-(-\omega, k)$ ) exchanges the poles of  $\Delta_+$  and  $\Delta_-$ :  $\Delta_+$  has two poles, one at positive  $\omega$ , with energy  $\omega_+(k)$ , and another one at negative

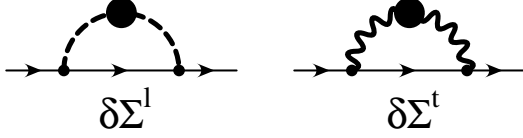


Fig. 1. NLO contributions to  $\delta\Sigma$  at hard momentum. Thick dashed and wiggly lines with a blob represent HTL-resummed longitudinal and transverse gauge boson propagators, respectively.

In contrast to the lowest order asymptotic mass  $M_\infty^2$ , the correction  $\delta M_\infty^2(k)$  is a nontrivial function of the momentum [3] which can be evaluated only numerically. However, this function contributes to eq. (10) only in an averaged form [2,3] :

$$\bar{\delta}M_\infty^2 = \frac{\int dk k [\partial f_+(k)/\partial\mu] \text{Re } 2k\delta\Sigma_+(\omega=k)}{\int dk k \partial f_+(k)/\partial\mu} = -\frac{1}{2\pi} g^2 C_f T \hat{m}_D, \quad (11)$$

where  $\hat{m}_D$ , the Debye mass, is

$$\hat{m}_D^2 = (2N + N_f) \frac{g^2 T^2}{6} + \frac{g^2}{2\pi^2} \sum_j \mu_j^2. \quad (12)$$

Thus, at a strictly perturbative level, it would be possible to reproduce the perturbative result for  $\mathcal{N}$  to order  $g^3$  by replacing  $2\hat{M}^2 \equiv M_\infty^2 \rightarrow M_\infty^2 + \bar{\delta}M_\infty^2$  in eqs. (7,8) for the HTL approximation to  $\mathcal{N}$ .

However, the correction  $\bar{\delta}M_\infty^2$  is negative, and for  $g \gtrsim 1$  it is of the same order of magnitude or larger than the lowest-order asymptotic mass, apparently leading to a tachyonic thermal mass. As we have argued previously [1,2,3], this problem is not specific to QCD, but can be studied already in simple scalar  $g^2\varphi^4$  theory. There the perturbative thermal mass to NLO is  $m^2 = g^2 T^2(1 - 3g/\pi)$ . The corresponding one-loop gap equation, on the other hand, gives a monotonic function of  $g$ , which is well approximated by the quadratic equation [3]  $m^2 = g^2 T^2 - 3mT/\pi$ . Also a simple Padé resummation [1]  $m^2 = g^2 T^2/(1 + 3g/\pi)$  gives reasonable approximations even for  $g \gg 1$ . In the following, we shall consider both prescriptions for including (11) into the asymptotic thermal mass, referring to them by “NLQ” and “NLP”, respectively.

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$\omega$ , with energy  $-\omega_-(k)$ ; these go over to  $\pm\hat{M}$  as  $k \rightarrow 0$ . Correspondingly,  $\Delta_-$  has poles at  $\omega_-$  and  $-\omega_+$ . See, e.g., Ref. [19].

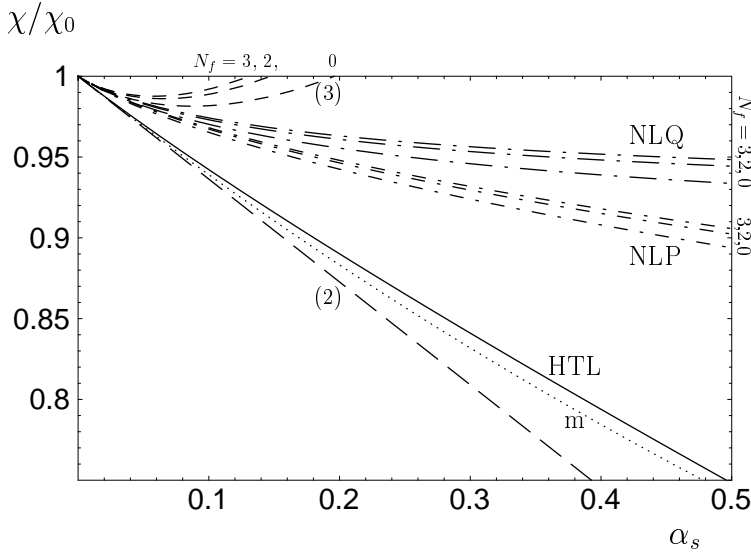


Fig. 2. Quark number susceptibility normalized to its free-field value in SU(3) with  $N_f = 0, 2, 3$  quark flavors as a function of  $\alpha_s$ . Dashed lines give the perturbative results through order  $\alpha_s$  (marked “(2)”) and order  $\alpha_s^{3/2}$  (marked “(3)”). The full line gives our result using HTL propagators, the dotted one a simpler quasiparticle model with momentum-independent mass  $M = M_\infty$  (both of which are  $N_f$  independent). Including next-to-leading order corrections to the asymptotic fermion mass through a quadratic gap equation gives the lines marked “NLQ”; using instead a simple Padé approximant gives the lines marked “NLP”.

### 3 Numerical evaluation

The results of a numerical evaluation of  $\chi/\chi_0$  are given in Fig. 2 as a function of  $\alpha_s$ .

The HTL approximation gives results which are above those of first-order perturbation theory (i.e., order  $g^2$ ). Since the former does not include anything of the plasmon effect  $\propto g^3$ , the visible deviation is to be attributed to higher order contributions. A numerical analysis reveals that most of the enhancement is due to terms of order  $g^4$ , which also involve a logarithm :

$$\chi_{HTL}^{(4)}|_{\mu=0} = N \left( 0.0431 \dots \times \log \frac{T}{\hat{M}} + 0.0028 \dots \right) \frac{\hat{M}^4}{T^2}. \quad (13)$$

The coefficient of the log is by a factor of  $\approx 0.52(N^2 - 1)/N^2$  smaller than that of the perturbative result (1); the complete coefficient would in fact be restored by  $O(g^4 \log(1/g)T^2)$  corrections to  $M_\infty^2$  (not considered here)<sup>2</sup>. Evidently, the

<sup>2</sup> The constant behind the logarithm (which is still unknown in perturbation theory) receives three-loop contributions which are beyond the  $\Phi$ -derivable two-loop approximation underlying the density expression (4).

HTL approximation incorporates already a sizable fraction of these higher-order terms.

It is instructive at this stage to compare with the susceptibility of an ideal gas of massive fermions with mass equal to the asymptotic HTL mass, and which therefore contains the correct contribution of order  $g^2$ . This reads (with  $\omega_k = \sqrt{k^2 + 2\hat{M}^2}$ ) :

$$\chi_{\text{m}}|_{\mu=0} = \frac{2}{T} \int_0^\infty \frac{dk k^2}{\pi^2} \frac{e^{\omega_k/T}}{(e^{\omega_k/T} + 1)^2} = 2 \frac{\partial}{\partial \log T} \int_0^\infty \frac{dk k^2}{\pi^2 \omega_k} \frac{1}{e^{\omega_k/T} + 1}. \quad (14)$$

In contrast to eq. (13), this does not involve any logarithmic term at order  $g^4$

$$\chi_{\text{m}}^{(4)}|_{\mu=0} = N \frac{7\zeta(3)}{4\pi^4} \frac{\hat{M}^4}{T^2} \approx 0.0216 N \frac{\hat{M}^4}{T^2}. \quad (15)$$

Numerically, however, (14) happens to be rather close to the HTL expression, as can be seen from the dotted line in Fig. 2.

In the self-consistent density, as we have seen, the (more decisive) order- $g^3$  contribution comes exclusively from the NLO correction to the asymptotic thermal mass. This introduces a (weak) dependence upon  $N_f$ , via the the Debye mass (12). As an estimate of this effect, we include it in the averaged form (11), for simplicity by a rescaling of  $\hat{M}$  for all momenta. In order to get an idea of the theoretical uncertainties, we do so alternatively through a quadratic gap equation (NLQ) or through a (2,1)-Padé approximant (NLP). The corresponding numerical results for  $N_f = 0, 2, 3$  are shown in Fig. 2 by the various dash-dotted lines, with the formal limit  $N_f = 0$  corresponding to the quenched approximation of lattice gauge theory. As manifest on this figure, the inclusion of the order- $g^3$  contribution in our self-consistent calculation has a significant effect, although not as dramatic as in conventional perturbation theory.

In Figs. 3 and 4 these numerical results are translated into plots of  $\chi/\chi_0$  as a function of  $T/T_c$  using the recent determination of  $T_c/\Lambda_{\overline{\text{MS}}}$  of Ref. [21] (which is found to differ significantly for quenched QCD and  $N_f = 2$ ), together with a standard two-loop running coupling  $\alpha_s(\bar{\mu})$ . We vary the renormalization scale  $\bar{\mu}$  around  $\bar{\mu} = 2\pi T$  by a factor of 2. For an error estimate of the NL approximations, we in addition combine the (overlapping) results for NLP and NLQ.

Also given in Figs. 3 and 4 are the recent lattice results of Refs. [8] and [9], respectively. These results involve finite but small quark masses, and, perhaps more importantly, are obtained for a lattice with only 4 sites in the temporal

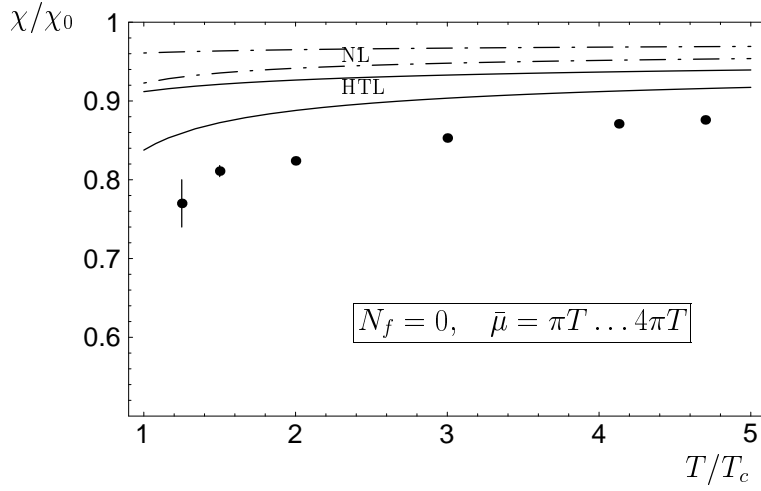


Fig. 3. Comparison of our results for  $\chi/\chi_0$  in massless QCD in the formal limit  $N_f = 0$  (using  $T_c/\Lambda_{\overline{\text{MS}}} = 1.15$  [21]) with the lattice results of Ref. [8] for quenched QCD obtained with quark mass  $m = 0.12T_c$  on a lattice with  $N_t = 4$  (no continuum extrapolation). [The two rightmost lattice data are unpublished, private communication by S. Gupta.]

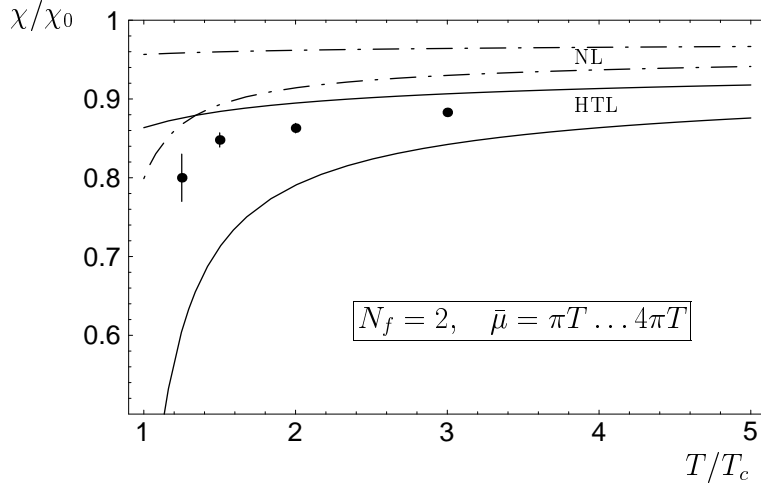


Fig. 4. Comparison of our results for  $\chi/\chi_0$  in massless  $N_f = 2$  QCD (using  $T_c/\Lambda_{\overline{\text{MS}}} = 0.49$  [21]) with the lattice results of Ref. [9] obtained with quark mass  $m = 0.1T_c$  on a lattice with  $N_t = 4$  (no continuum extrapolation).

direction, and are still waiting for a proper continuum extrapolation. Our results follow the same general trend as the lattice data (they slowly increase towards the ideal gas value), but exceed the latter by some +10%. (Remarkably, this discrepancy is less pronounced for the physical case of dynamical fermions.) By contrast, the perturbative result to order  $g^3$ , eq. (1), decreases as a function of  $T/T_c$  in the range studied here and, actually, up to temperatures as high as  $T \sim 100T_c$  for  $N_f = 0$ , and even higher for  $N_f = 2$ .



## 4 Off-diagonal susceptibilities

The systematics of the diagrammatic contributions to susceptibilities (in particular, the off-diagonal ones) can be clarified by referring to the symmetry under charge conjugation, or  $C$ -parity. Chemical potentials couple to the fermion fields in the same way as the  $A^0$  component of an Abelian gauge field. Thus, when expanding a quark loop in powers of  $\mu$ , one may attribute to each factor of  $\mu$  the  $C$ -parity of the photon field, i.e.,  $C = -1$ . Gluons attached to a quark loop in a colour symmetric state behave under permutations in the same way as photon insertions, and thus can be ascribed  $C = -1$  as well.  $C$ -parity conservation forbids a photon to decay into two gluons: a colourless 2-gluon state is necessarily colour symmetric, and therefore  $C$ -even. However, a photon can decay into three gluons which are in a colour symmetric state, or in two gluons and an arbitrary odd number of photons, etc. In terms of chemical potentials, this means that a quark loop with two gluon external lines is necessarily even in  $\mu$ , while a quark loop with three gluon legs may generate also a term linear in  $\mu$ , which is then symmetric in the colour indices.

The first perturbative contributions to the nondiagonal susceptibility  $\tilde{\chi}$  require two fermion loops connected by gluon lines. The diagram with just one gluon exchange vanishes by colour neutrality. The one with two gluon exchange is non-zero, but because the fermion loops are then even functions of  $\mu$ , it contributes to  $\tilde{\chi}$  only when  $\mu \neq 0$ , starting at order  $g^3$ . One easily finds

$$\chi_{ij} = \frac{g^4(N^2 - 1)T\mu_i\mu_j}{16\pi^5 m_D} \quad \text{for } i \neq j. \quad (16)$$

In fact, this is the same as  $\chi_{ij} = \partial \mathcal{N}_i^{(3)} / \partial \mu_j$  ( $i \neq j$ ) with  $\mathcal{N}^{(3)}$  given in eq. (10). From the perspective of eq. (10), the mixing between different flavors is induced by the resummation of quark loops along the soft, internal, gluon lines in the diagrams in Fig. 1.

However, when all chemical potentials vanish, the lowest-order diagram contributing to  $\tilde{\chi}$  is the “bugblatter” [22] diagram shown in Fig. 5a.

This diagram is superficially of order  $g^6$ , but when calculated with bare gluon propagators, it develops a logarithmic infrared divergence in the electrostatic sector, because, for static external gluons, the quark loop induces an effective local vertex corresponding to

$$\frac{1}{3!} \text{Tr } A_0^3 \sum_i \frac{\partial^3}{\partial \mu_i^3} P_0^f(m_i; \mu_i, T) \quad (17)$$

where  $P_0^f$  is the ideal gas pressure of a fermion with mass  $m_i$  and chemical

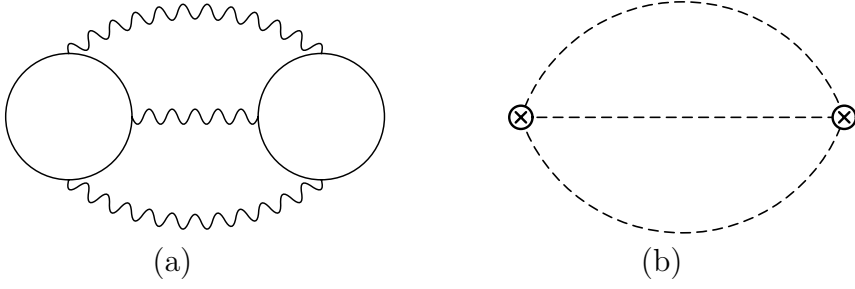


Fig. 5. (a) Lowest-order diagram in the thermodynamic potential that contributes to off-diagonal susceptibilities  $\partial \mathcal{N}_i / \partial \mu_j$ . (b) Corresponding diagram in the effective theory for the electrostatic modes.

potential  $\mu_i$ . For massless fermions this reduces to [23,24,25] :

$$\frac{g^3}{3\pi^2} \text{Tr} A_0^3 \sum_i \mu_i = \frac{g^3}{12\pi^2} d^{abc} A_0^a A_0^b A_0^c \sum_i \mu_i \quad (18)$$

( $A_\mu = A_\mu^a t^a$ ,  $\text{Tr} t^a t^b = \delta^{ab}/2$ ). We expect Debye screening to cut off this divergence at the scale  $\hat{m}_D \sim gT$ , with the upper scale in the logarithm, of order  $T$ , set by the thermal distribution. This gives a contribution to  $\tilde{\chi}$  of order  $g^6 \log(1/g)$ , which is the leading order effect for  $g$  small enough. We now compute its coefficient.

In the imaginary time formalism, the infrared divergence is isolated in the static Matsubara sector. The original diagram in Fig. 5a is then identified with the two-loop diagram in Fig. 5b. Formally, this is the second order perturbative correction  $f_2$  to the free energy  $f = -\log Z$  of a 3-dimensional scalar field with effective action

$$S_E = \int d^3x \frac{1}{2} A_0^a (-\nabla^2 + \hat{m}_D^2) A_0^a + i \sum_j \mu_j \frac{g^3 \sqrt{T}}{12\pi^2} d^{abc} A_0^a A_0^b A_0^c, \quad (19)$$

where  $A_0^a(\mathbf{x}) = \sqrt{T} A_0^a(\omega_n = 0, \mathbf{x})$ ,  $\hat{m}_D$  is the Debye mass (12), and the interaction term is now purely imaginary, as a consequence of the continuation  $A_0^M \rightarrow iA_0^E$  to imaginary time. Denoting by  $S_I$  the interaction term in eq. (19), we have  $f_2 = -\langle S_I^2/2 \rangle_0$ , which is *positive*, a consequence of the interaction term being purely imaginary. A direct calculation yields:

$$\frac{f_2}{V} = 3d^{abc}d^{abc} \left( \sum_j \mu_j \frac{g^3 \sqrt{T}}{12\pi^2} \right)^2 \int \frac{d^3k d^3q}{(2\pi)^6} D_{00}(k) D_{00}(q) D_{00}(|\mathbf{k} + \mathbf{q}|), \quad (20)$$

where  $D_{00} = 1/(k^2 + \hat{m}_D^2)$  and  $d^{abc}d^{abc} = (N^2 - 1)(N^2 - 4)/N$ . The above integral has a spurious ultraviolet divergence, which comes from the restriction

to the static Matsubara modes, and, in the absence of the Debye mass, it would be also divergent in the infrared. This yields

$$\int_0^\Lambda \frac{d^3k \, d^3q}{(2\pi)^6} \frac{1}{[k^2 + \hat{m}_D^2][q^2 + \hat{m}_D^2][(\mathbf{k} + \mathbf{q})^2 + \hat{m}_D^2]} \simeq \frac{1}{16\pi^2} \log \frac{\Lambda}{\hat{m}_D} \simeq \frac{1}{16\pi^2} \log \frac{1}{g}, \quad (21)$$

where the upper cut-off  $\Lambda$  eventually gets replaced by  $T$  upon inclusion of the nonstatic Matsubara modes.

Putting everything together and returning to the original 4-dimensional gauge theory at finite temperature, the above estimate for  $f_2$  translates into the following, negative, contribution to the pressure:

$$\Delta P = -\frac{T}{V} f_2 = -\frac{(N^2 - 1)(N^2 - 4)}{768N} T^2 \left( \sum_j \mu_j \right)^2 \left( \frac{g}{\pi} \right)^6 \log \frac{1}{g}. \quad (22)$$

From eq. (22) we finally deduce the leading-order term for nondiagonal  $\chi$ :

$$\frac{\tilde{\chi}}{\chi_0} \simeq -\frac{(N^2 - 1)(N^2 - 4)}{128N^2} \left( \frac{g}{\pi} \right)^6 \log \frac{1}{g}. \quad (23)$$

This logarithmically enhanced contribution vanishes in SU(2) gauge theories—though not in QED. The (ultrarelativistic) QED result is obtained by replacing  $d^{abc}d^{abc} \rightarrow 16$  in eq. (20) (cf. eq. (18)), and  $\chi_0 \rightarrow T^2/3$ , yielding

$$\tilde{\chi}/\chi_0|_{\text{QED}} \simeq -\frac{e^6}{8\pi^6} \log(1/e) \simeq -4(\alpha/\pi)^3 \log(1/\alpha). \quad (24)$$

In QCD, the leading-order contribution to  $\tilde{\chi}$  is

$$\frac{\tilde{\chi}}{\chi_0}|_{N=3} \simeq -\frac{5}{144} \left( \frac{g}{\pi} \right)^6 \log \frac{1}{g} \simeq -\frac{10}{9\pi^3} \alpha_s^3 \log(1/\alpha_s) \quad (25)$$

While in QED it is plausible that the leading-log term with its negative coefficient dominates over other contributions  $\propto \alpha^3$ , this is more uncertain in QCD, where  $\log(1/\alpha_s)$  is much smaller. But assuming that the unknown constant behind the log is roughly of order 1, the off-diagonal susceptibilities appear to be rather tiny in (massless) QCD (although they would tend to become more important for larger  $N$ ). In fact, a most recent lattice study of nondiagonal susceptibilities in  $N_f = 2$  QCD [9] has found only values consistent with zero,

but within statistical errors that are  $\lesssim 10^{-6}$ , whereas the natural order of magnitude of (25) is given by  $\frac{10}{9\pi^3}[\alpha_s(2\pi T)]^3 \sim 10^{-4}$  for  $T \sim 3T_c$ .

The lattice calculations in Ref. [9] have been performed with finite quark masses down to  $m/T_c = 0.1$ . However, this should not lead to any noticeable reduction, because the first  $m/T$  correction in (17) is only quartic, leading to a correction factor  $\approx (1 - 0.06187m^4/T^4)$  in the final result (25), so the extreme smallness of the lattice result of Ref. [9] remains a mystery for now.

## 5 Conclusions

To summarize, we have presented an analytical calculation of the diagonal quark number susceptibility in hot QCD within an approximately self-consistent resummation of perturbation theory. Our (non-perturbative) formulae include completely the perturbative contributions of  $\mathcal{O}(g^2)$  and  $\mathcal{O}(g^3)$ , together with a sizable fraction (of, roughly, 50%) of the contribution of order  $g^4 \log(1/g)$ . For temperatures between 1.5 to  $5T_c$ , our results show the same general trend as seen on the lattice – namely, a slow increase towards the ideal gas results from below –, but with absolute values which are slightly, but systematically, above the lattice data, by 5–10% in the case of  $N_f = 2$ . This deviation is somewhat larger than that of our analogous calculations of the entropy density [1,2,3]. However, given that a continuum extrapolation of the lattice data for quark susceptibilities is still missing, it remains to be seen whether there are sizable higher-order perturbative contributions not captured by our approach or even important non-perturbative phenomena, as speculated in Ref. [9].

We have further computed the off-diagonal susceptibility to lowest non-trivial order in perturbation theory, that is, to order  $g^6 \log(1/g)$ . The result turns out to be remarkably small, but not so small, however, to explain the corresponding lattice result of Ref. [9], which, surprisingly, is consistent with zero with statistical errors  $\lesssim 10^{-6}$ . This discrepancy certainly calls for more investigations and more lattice data.

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